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Random field Ising systems on a general hierarchical lattice: Rigorous inequalities

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Random Ising systems on a general hierarchical lattice with both random fields and random bonds are considered. Rigorous inequalities between eigenvalues of the Jacobian renormalization matrix at the pure fixed point are obtained. These inequalities lead to upper bounds on the crossover exponents $\{\phi_i\}$.

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Despite the many years of research and the large number of researchers working on the subject, the study of critical behavior of random systems has led to only few exact results. On the other hand, some of these results [1,2] played an important role, especially in the context of the random field problem. In a recent study [3] we considered a random bond Ising system on a general hierarchical lattice, where the renormalization group (RG) transformation is exact [4], and obtained inequalities concerning the eigenvalues $\{\lambda_i\}$ of the Jacobian renormalization matrix, at the pure fixed point. The purpose of the present study is to show that similar inequalities can be obtained if random fields are included. In contrast to the case of random bonds and zero fields, correlations are now generated by the renormalization flow. Nevertheless, it appears that these correlations are, first, confined to the fields so that the distribution of bonds is left uncorrelated, and second, restricted to nearest-neighbor (NN) correlations. It is important to emphasize that these short-ranged field correlations are generated by the RG transformation even if one assumes no correlations to begin with, and that the range of correlations does not increase under the transformation. Our results are relevant to real lattices, since some approximate RG schemes on real lattices are in fact exact RG schemes on hierarchical lattices (Migdal-Kadanoff [5,6] and others [7]) and since it is believed that the critical behavior of an Ising system on a real lattice can be mimicked by that behavior on a properly chosen hierarchical lattice [8–11].

We consider a general hierarchical lattice described schematically in Fig. 1. The shaded area shown in (a) consists of a set of lattice points where some of the pairs are joined. In (b), a typical shaded area is represented. The solid lines are bonds to be iterated in constructing the lattice while the dashed ones are not to be iterated. The bold lines represent the possibility for some bonds to be strengthened, namely, multiplied by some constant. The random Ising system is represented by the dimensionless Hamiltonian

$$-\mathcal{H}=\sum_{(i,j)} J_{ij}\sigma_i\sigma_j+\sum_i h_i\sigma_i, \tag{1}$$

where (i,j) refers to connected pairs only. All three types of bonds of Fig. 1(a) then carry a coupling $J^{12}_{\alpha\beta}$ (for the bonds joining sites α and β), while each site carries a field h^{12}_{α}. (Note that one of the members of the pair $\alpha\beta$ may be either 1 or 2.)

The renormalized couplings and renormalized fields are given by

$$\tilde{J}_{12}=f_J\{J^{12}_{\alpha\beta},h^{12}_{\alpha}\} \tag{2a}$$

and

$$\tilde{h}_i=h_i+\sum_{j=1}^{\tilde{z}_i} f_h\{J^{12}_{\alpha\beta},h^{12}_{\alpha}\}, \quad i=1,2, \tag{2b}$$

respectively, while $\tilde{z}_i$ is the coordination number of the site i on the renormalized lattice. Both, f_J and f_h, depend only on couplings and fields within the rescaling volume associated with the pair of sites (1,2) [the shaded area in Fig. 1(a)]. Equation (2a) implies that $\tilde{J}_{ij}$ and $\tilde{J}_{lm}$ are not correlated if the pairs (i,j) and (l,m) are not identical. This does not hold for the renormalized fields. Due to the sum in Eq. (2b) over NN sites on the renormalized lattice, it is clear that even if there are no correlations to begin with, correlations are generated by the RG transformation, between fields on NN sites and fields and couplings on a site and a bond attached to it. [For example, in Fig. 1(b), the following pairs are correlated:

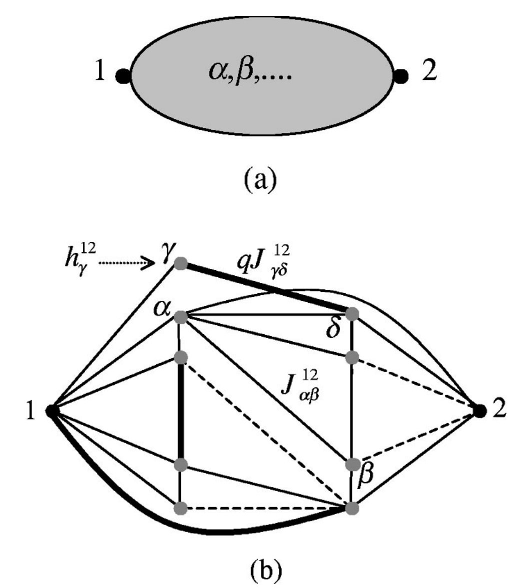

FIG. 1. A general hierarchical lattice is described schematically. In (a), the shaded area consists of a set of lattice points, $\alpha,\beta,\ldots,$ where some of the pairs are joined. In (b), a typical shaded area is represented. The solid lines are bonds to be iterated in constructing the lattice, while the dashed ones are not to be iterated. The bold lines represent the possibility for some bonds to be strengthened, multiplied by some constant.

$(\tilde{h}_1,\tilde{h}_2)$, $(\tilde{J}_{12},\tilde{h}_1)$, and $(\tilde{J}_{12},\tilde{h}_2)$.] It is easier to deal with such correlations by considering a bond-field Hamiltonian of the form

$$-\mathcal{H}=\sum_{(i,j)}[J_{ij}\sigma_i\sigma_j+h_{ij}(\sigma_i+\sigma_j)], \tag{3}$$

in which the random variables are the couplings J_{ij} and the bond fields h_{ij}. The set of RG transformation equations is now given by

$$\tilde{J}_{12}=f_J\{J^{12}_{\alpha\beta},h^{12}_{\alpha\beta}\} \tag{4a}$$

and

$$\tilde{h}_{12}=f_h\{J^{12}_{\alpha\beta},h^{12}_{\alpha\beta}\}. \tag{4b}$$

Equations (4), therefore imply that none of the two couplings $\tilde{J}_{ij}$ or $\tilde{h}_{ij}$, is correlated with any of the two couplings $\tilde{J}_{lm}$ or $\tilde{h}_{lm}$, if the pairs (i,j) and (l,m) are not identical.

In terms of the bond fields, the site fields are given by

$$h_i=\frac{1}{2}\sum_{j(i)}h_{ij}, \tag{5}$$

where $j(i)$ indicates that the sum is over all sites j connected to i. A similar bond-field Hamiltonian (3) was already used in the past [12–14], only with the additional term $\Sigma_{(i,j)}h^{\dagger}_{ij}(\sigma_i-\sigma_j)$. Note that it is necessary to include the dagger fields only if one assumes that the site fields are initially uncorrelated. Since, however, the initial state of non-correlated site fields is not preserved by RG transformation and nn correlations between site fields are developed, there is no reason to start with uncorrelated fields on the sites.

We assume the existence of a ferromagnetic fixed point at $\{J_{\alpha\beta}\}=J^*$ and $\{h_{\alpha\beta}\}=0$. We denote the departure of $J_{\alpha\beta}$ from J^* by $\delta J_{\alpha\beta}$ and define the moments as

$$\Gamma_{ij}=\langle(\delta J_{\alpha\beta})^i(h_{\alpha\beta})^j\rangle. \tag{6}$$

Clearly at the fixed point $\Gamma^*_{ij}=0$. We will be interested in the recursion relations of the moments near the pure fixed point,

$$\tilde{\Gamma}_{ij}=G_{ij}\{\Gamma_{lm}\}. \tag{7}$$

The recursion relations above are obtained from the recursion relations for the local couplings

$$\delta\tilde{J}_{12}=\sum_{(\alpha,\beta)}\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\delta J_{\alpha\beta}+\frac{1}{2}\sum_{(\alpha,\beta)}\left(\frac{\partial^2 f_J}{\partial J^2_{\alpha\beta}}\right)^*(\delta J^2_{\alpha\beta})+\frac{1}{2}\sum_{(\alpha,\beta)}\left(\frac{\partial^2 f_J}{\partial h^2_{\alpha\beta}}\right)^*h^2_{\alpha\beta}+\cdots \tag{8a}$$

and

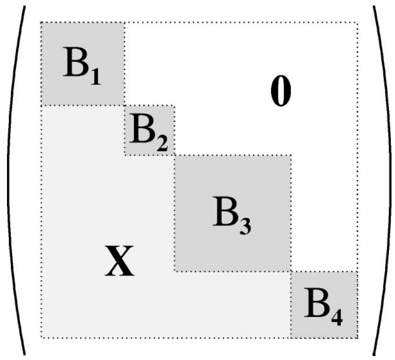

FIG. 2. A general schematic description of a ''block-triangular'' matrix is shown. $\mathbf{B}_1,\ldots,\mathbf{B}_4$ represent the blocks along the main diagonal. The gray area marked with **X** indicates the presence of nonzero matrix elements while in the area marked with 0, all elements are zero.

$$\tilde{h}_{12}=\sum_{(\alpha,\beta)}\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*h_{\alpha\beta}+\sum_{(\alpha,\beta)}\left(\frac{\partial^2 f_h}{\partial J_{\alpha\beta}\partial h_{\alpha\beta}}\right)^*\delta J_{\alpha\beta}h_{\alpha\beta}+\cdots, \tag{8b}$$

where $(\cdots)^*$ denotes evaluation at the pure fixed point. Note that although we are interested only in the expansion of $\tilde{\Gamma}_{ij}$ to first order in Γ_{lm}, we still need, in principle, orders higher than linear in Eqs. (8) above. Note also the terms missing in the expansions due to the different parities of J and h. The renormalized moments are given by

$$\tilde{\Gamma}_{ij}=\sum_{(\alpha,\beta)}\left[\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\right]^i\left[\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*\right]^j\Gamma_{ij}+\sum_{(\alpha,\beta)}\sum_{l,m}A^{ij}_{lm}\Gamma_{lm}, \tag{9}$$

where clearly, in the last sum on the right-hand side of the above, $l+m>i+j$. Also the parity of m in the sum must equal the parity of j. The A^{ij}_{lm}'s with $l+m>i+j$ always involve derivatives higher than the first of at least one of the f's. We arrange next the Γ_{ij}'s using a single index

$$\mathcal{G}_k=\Gamma_{ij}, \tag{10}$$

with

$$k=\frac{(i+j)(i+j+1)}{2}+j+1. \tag{11}$$

This brings Eq. (9) into the standard matrix notation

$$\tilde{\mathcal{G}}_m=A_{mn}\mathcal{G}_n. \tag{12}$$

It is not difficult to show that if k_1 corresponds to (i,j) and k_2 to (l,m) then $l+m>i+j$ implies $k_2>k_1$. This means that the matrix A is block-triangular (Fig. 2). Consider next one of the blocks along the diagonal of A. From the expansions (8) and (9) it follows directly that the only contribution to $(\delta\tilde{J})^i(\tilde{h})^j$ in $(\delta J)^l(h)^m$ such that $l+m=i+j$, is the one with $l=i$ and $m=j$. The final conclusion thus is that the matrix A

is triangular, so that its eigenvalues are just its diagonal terms. The eigenvalues of the Jacobian transformation matrix are thus

$$\lambda_{ij}=\sum_{(\alpha,\beta)}\left[\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\right]^i\left[\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*\right]^j. \tag{13}$$

This leads now to a number of interesting inequalities.

(a) All eigenvalues are positive,

$$\lambda_{ij}\geqslant 0. \tag{14a}$$

(b) All eigenvalues are ordered,

$$\lambda_{i+1,j}<\lambda_{ij} \quad \text{and} \quad \lambda_{i,j+1}<\lambda_{ij}. \tag{14b}$$

(c) All eigenvalues obey a convexity condition,

$$\lambda_{ij}\lambda_{kj}\geqslant\lambda_{i+k,j} \quad \text{and} \quad \lambda_{ij}\lambda_{ik}\geqslant\lambda_{i,j+k}. \tag{14c}$$

(d) All eigenvalues obey

$$(\lambda_{ij})^2\leqslant\lambda_{i+k,j+l}\lambda_{i-k,j-l}, \tag{14d}$$

where, in (d), $k=-i,\ldots,i$ and $l=-j,\ldots,j$.

Proof: In a recent paper [3] we have considered random bond Ising systems for which the subset $\{\lambda_{i0}\}$ is considered. There, we have already proven properties (a)–(c) and our proof here will follow the same line.

Properties (a) and (c) are proven by showing that

$$\frac{\partial f_J}{\partial J_{\alpha\beta}}(J^*,0)\geqslant 0 \tag{15a}$$

and

$$\frac{\partial f_h}{\partial h_{\alpha\beta}}(J^*,0)\geqslant 0. \tag{15b}$$

We have to consider then the specific transformations generated by

$$-\tilde{\mathcal{H}}=\ln \mathrm{tr}' e^{-\mathcal{H}}, \tag{16}$$

where tr′ represents trace only over the subset of spins $\{\sigma_\alpha\}$ internal to the rescaling volume, not including the external spins σ_1 and σ_2. The renormalized couplings and fields given by Eqs. (4), can now be written in the forms

$$\tilde{J}_{12}=-\tfrac{1}{4}\mathrm{tr}_{12}[\sigma_1\sigma_2\tilde{\mathcal{H}}], \tag{17a}$$

$$\tilde{h}_{12}=-\tfrac{1}{4}\mathrm{tr}_{12}[(\sigma_1+\sigma_2)\tilde{\mathcal{H}}], \tag{17b}$$

where tr_{12} indicates trace over the two external spins σ_1 and σ_2. The derivatives of $\tilde{J}_{12}$ with respect to $J_{\alpha\beta}$ and $\tilde{h}_{12}$ with respect to $h_{\alpha\beta}$ are thus given by

$$\frac{\partial\tilde{J}_{12}}{\partial J_{\alpha\beta}}(J^*,0)=\frac{1}{4}\,\mathrm{tr}_{12}(\sigma_1\sigma_2)\langle\sigma_\alpha\sigma_\beta\rangle_{12} \tag{18a}$$

and

$$\frac{\partial\tilde{h}_{12}}{\partial h_{\alpha\beta}}(J^*,0)=\frac{1}{4}\,\mathrm{tr}_{12}(\sigma_1+\sigma_2)\langle\sigma_\alpha+\sigma_\beta\rangle_{12}, \tag{18b}$$

where $\langle\cdots\rangle_{12}$ is the average with respect to $\mathcal{H}$ with σ_1 and σ_2 held fixed. In calculating the above derivatives at the pure fixed point, we use the following symmetry properties of the system:

$$\langle\sigma_\alpha\sigma_\beta\rangle^*_{++}=\langle\sigma_\alpha\sigma_\beta\rangle^*_{--}, \tag{19a}$$

$$\langle\sigma_\alpha\sigma_\beta\rangle^*_{+-}=\langle\sigma_\alpha\sigma_\beta\rangle^*_{-+}, \tag{19b}$$

$$\langle\sigma_\alpha\rangle^*_{++}=-\langle\sigma_\alpha\rangle^*_{--}, \tag{19c}$$

$$\langle\sigma_\alpha\rangle^*_{+-}=-\langle\sigma_\alpha\rangle^*_{-+}, \tag{19d}$$

to obtain

$$\frac{\partial\tilde{J}_{12}}{\partial J_{\alpha\beta}}(J^*,0)=\frac{1}{2}[\langle\sigma_\alpha\sigma_\beta\rangle^*_{++}-\langle\sigma_\alpha\sigma_\beta\rangle^*_{+-}] \tag{20a}$$

and

$$\frac{\partial\tilde{h}_{12}}{\partial h_{\alpha\beta}}(J^*,0)=\frac{1}{2}[\langle\sigma_\alpha+\sigma_\beta\rangle^*_{++}]. \tag{20b}$$

Here the sign indices specifically indicate the state of the spins σ_1 and σ_2 and the * indicates that the average is with respect to the pure fixed point Hamiltonian

$$-\mathcal{H}^*=J^*\sum_{(i,j)}\sigma_i\sigma_j. \tag{21}$$

Now, according to the GKS inequalities [15,16], if all the many-spin couplings $J_A=h_\alpha,J_{\alpha\beta},\ldots$ in a general Ising system are positive, all the many-spin correlations $\langle\sigma_A\rangle=\langle\sigma_\alpha\rangle,\langle\sigma_\alpha\sigma_\beta\rangle,\ldots$ must obey $\langle\sigma_A\rangle\geqslant 0$. Using Eqs. (20), the averages are taken with respect to the pure ferromagnetic Hamiltonian (21), where the two external spins of each of the rescaling volumes, which are held fixed, serve effectively as local fields. When these effective fields are held both positive, the GKS inequalities hold, so that

$$\langle\sigma_\alpha\rangle^*_{++}\geqslant 0 \quad \text{and} \quad \langle\sigma_\alpha\sigma_\beta\rangle^*_{++}\geqslant 0, \tag{22}$$

which is enough already to prove inequality (15b). Inequality (15a) can be easily shown to hold using, in addition to the GKS inequalities, other rigorous inequalities, just recently proven [17], also concerning the many-spin correlations in general Ising systems. It states that if all the many-spin couplings J_A are positive again, the absolute value of all the many-spin correlations $\langle\sigma_A\rangle$ does not increase when the value of any of the couplings is reduced, taking any value in the interval $[-J_A,J_A]$. According to this, it is clear that under reversal of the $+1$ state of any of the two external spins, the many-spin correlations cannot increase. So, we arrive at the conclusion that

$$\langle\sigma_\alpha\sigma_\beta\rangle^*_{++}\geqslant\langle\sigma_\alpha\sigma_\beta\rangle^*_{+-} \tag{23}$$

and (15a) is also proven. This completes the proof of properties (a) and (c).

We turn now to property (b). Here we need to show that

$$\frac{\partial f_J}{\partial J_{\alpha\beta}}(J^*,0)<1 \tag{24a}$$

and

$$\frac{\partial f_h}{\partial h_{\alpha\beta}}(J^*,0)<1. \tag{24b}$$

at any finite temperature. But, referring to Eqs. (20) again, it is clear that

$$\tfrac{1}{2}[\langle\sigma_\alpha\sigma_\beta\rangle^*_{++}-\langle\sigma_\alpha\sigma_\beta\rangle^*_{+-}]\leqslant\tfrac{1}{2}[|\langle\sigma_\alpha\sigma_\beta\rangle^*_{++}|+|\langle\sigma_\alpha\sigma_\beta\rangle^*_{+-}|]\leqslant 1 \tag{25a}$$

and that

$$\tfrac{1}{2}[\langle\sigma_\alpha\rangle^*_{++}+\langle\sigma_\beta\rangle^*_{++}]\leqslant\tfrac{1}{2}[|\langle\sigma_\alpha\rangle^*_{++}|+|\langle\sigma_\beta\rangle^*_{++}|]\leqslant 1, \tag{25b}$$

while the equality sign can hold only at zero temperature. This proves property (b).

We are left now with property (d). Here we use the more general definition of a scalar product, $(\mathbf{u},\mathbf{v})\equiv\Sigma_i w_i u_i^* v_i$ where $\forall i,\ w_i\geqslant 0$ and the corresponding Schwartz inequality, which reads $(\Sigma_i w_i u_i^* v_i)^2\leqslant\Sigma_i w_i|u_i|^2\Sigma_j w_j|v_j|^2$ (here is the only place where the * represents complex conjugate). We replace, next, the sum over the single index i with the double index $(\alpha\beta)$ and identify

$$u_{\alpha\beta}\equiv\left[\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\right]^r\left[\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*\right]^s \tag{26a}$$

with $r=0,\ldots,i$ and $s=0,\ldots,j$,

$$v_{\alpha\beta}\equiv\left[\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\right]^p\left[\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*\right]^q \tag{26b}$$

with $p=0,\ldots,i-r$ and $q=0,\ldots,j-l$, and

$$w_{\alpha\beta}\equiv\left[\left(\frac{\partial f_J}{\partial J_{\alpha\beta}}\right)^*\right]^{i-r-p}\left[\left(\frac{\partial f_h}{\partial h_{\alpha\beta}}\right)^*\right]^{j-s-q}, \tag{26c}$$

to obtain

$$(\lambda_{ij})^2\leqslant\lambda_{i+r-p,j+s-q}\lambda_{i-r+p,j-s+q}, \tag{27}$$

where we have used the fact that all partial derivatives are real and positive. All that is left now is to denote $r-p=k$ with $k=-i,\ldots,i$ and, similarly, $s-q=l$ with $l=-j,\ldots,j$, which completes the proof of property (d).

In addition to that, denoting by $m_J<1$ and $m_h<1$, the maximal values of $\partial\tilde{J}_{ij}/\partial J_{\alpha\beta}$ and $\partial\tilde{h}_{ij}/\partial h_{\alpha\beta}$, respectively, we obtain

$$\lambda_{ij}\leqslant\lambda_{kl}m_J^{i-k}m_h^{j-l},\quad\text{with}\quad k=0,\ldots,i\quad\text{and}\quad l=0,\ldots,j, \tag{28}$$

so that we have also proven that the number of relevant interactions at the pure fixed point is finite. The only case of which the equality sign holds is the diamond hierarchical lattice (DHL) [4,18], where all bonds are equivalent. From (28) follows an inequality for the crossover exponents:

$$\phi_{ij}<1+\frac{(i-1)\ln m_J+j\ln m_h}{\ln\lambda_{10}}<1,\quad i+j=2,3,\ldots, \tag{29}$$

where $\phi_{ij}=y_{ij}/y_{10}$, $y_{ij}=\ln\lambda_{ij}/\ln b$, and b is the rescaling factor. The condition for criticality of the pure fixed point is $\max(\lambda_{20},\lambda_{11},\lambda_{02})<1$, while else, we expect a random critical point with a different set of critical exponents. It is interesting to note that it was just recently shown [19] that even for the random bond Ising system, the Harris criterion for pure criticality [20], $\alpha_p<0$ (α_p being the specific heat exponent) is equivalent to the obvious requirement, $\lambda_{20}<1$, only in the special case of the DHL. In the more general case, it was shown that $\alpha_p\leqslant\phi_{20}$ so that the Harris criterion is only a necessary condition for pure criticality to hold and counter examples where $\alpha_p<0$ and $\phi_{20}>0$ have been presented. The analogous result for the random system is $\gamma_p\leqslant\phi_{02}$ (γ_p being the susceptibility exponent of the pure system) but since γ_p turns out always to be positive, the random field is always relevant at the pure critical point.

We wish to conclude by emphasizing that the inequalities proven here hold not only for exact RG transformations on HL's but also for all other renormalization schemes (such as MK scheme [5,6]) in which the renormalized couplings or fields are not correlated or even in cases where it is clear that the correlations are not important [21].

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